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Self-preserving flow inside a turbulent boundary layer

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If a thick, turbulent boundary layer is disturbed near the rigid boundary, the flow changes are confined initially to a thin layer adjacent to the boundary. Elliott (1958) and Panofsky & Townsend (1964) have attempted to calculate the flow disturbance caused by an abrupt change in surface roughness by assuming special velocity distributions which are consistent with a logarithmic velocity variation near the boundary. Inspection of their distributions shows that the deviations from the upstream distribution are self-preserving in form, and it is shown that self-preserving development is dynamically possible if $\log l_0/z_0$ (l_0 being depth of modified flow, z_0 roughness length) is fairly large and if l_0 is small compared with the total thickness of the layer. Other kinds of surface disturbance may lead to self-preserving changes of the original flow and the theory is developed also for flow downstream of a line roughness, for the temperature distribution downstream of a boundary separating an upstream region of uniform roughness and heat-flux from a region of different or possibly varying roughness and heat-flux, and for the return of a complete boundary layer to self-preserving development after a disturbance. The requirement that the distributions of velocity and temperature should conform to the logarithmic, equilibrium forms near the surface makes the predictions of surface stress and surface flux nearly independent of the exact nature of the turbulent transfer process, and the profiles of velocity and temperature are determined within narrow limits by the surface fluxes. To provide explicit profiles, the mixing-length transfer relation is used. Its validity for the self-preserving flows is discussed in an appendix.

1. Introduction

Experience shows that a turbulent flow which is capable of developing in a self-preserving way does so to a good approximation, well-known examples being jets, wakes and boundary layers. The importance of self-preserving flow is that rates of change of velocity and length scales can be predicted with no more specific assumption about the nature of turbulent motion than that the large-scale motion is independent of the fluid viscosity. Whether a flow can be self-preserving or not depends on the boundary conditions, an adequate test being whether the Reynolds equations for the mean velocity and the turbulent energy can be satisfied to a fair approximation by self-preserving distributions of mean velocity, Reynolds stress, turbulent energy, etc., which satisfy the boundary conditions. For a 'simple' flow such as a wake, self-preservation means that the lateral distributions of each quantity have the same form at all distances from a flow origin, differing only in common scales of velocity and length. In particular, the entrainment velocity (the velocity with which turbulent flow spreads into the ambient fluid) is proportional to the velocity scale of the mean flow. The flows considered here are perturbations of either equilibrium flows or simply self-preserving flows, and it is the deviation of a flow quantity from its value in the undisturbed flow that has a self-preserving form. The test for the possibility of self-preserving development is the same, but, to satisfy it, different scales of velocity must be used for the deviations of mean velocity and turbulent stress. As a result, the 'entrainment velocity' (now defined as the velocity with which the flow perturbation spreads inside the basic flow) is set by the basic flow and is independent of the magnitude of the perturbation.

These self-preserving flows are most likely to be met in the earth's boundary layer and the simplest one is found when a boundary layer passes from a surface of one roughness to a surface of different roughness, a problem treated by Elliott (1958) and by Panofsky & Townsend (1964), each assuming self-preservation of the velocity changes. The more general approach provides the justification for the assumption of self-preserving development and shows that flow changes are substantially independent of special assumptions about the interaction between the turbulence and the mean flow. In this paper, I consider self-preserving development of velocity and temperature perturbations of a deep boundary layer and of velocity over a complete boundary layer. In another paper, the theoretical results will be developed in forms suitable for practical applications and the predictions compared with some of the available observations.

2. The perturbation flow in a deep boundary layer

Consider a deep, turbulent boundary layer caused by flow in the Ox-direction over a horizontal surface with negligible pressure gradient. Vertical transfer of heat is assumed to be too small to affect the turbulent motion. For negative x, the surface is uniformly rough, and the boundary layer is assumed to have attained the equilibrium self-preserving form and a total thickness large compared with any of the heights considered. Then upstream of $x = 0$, the Reynolds stress is nearly independent of height and is equal to the surface stress† u_1^2, and the vertical distribution of mean velocity is given by

$$U_1 = \frac{u_1}{k} \log \frac{z}{z_1}, \tag{2.1}$$

where k is the Karman constant, u_1 is the friction velocity, z is the height, z_1 is the roughness length of the surface for $x < 0$. For positive values of x, the surface roughness varies and causes modification of the flow. Since the rate of production of turbulent energy in the equilibrium layer upstream is $u_1^3/(kz)$ (also equal to the rate of energy dissipation) and the kinetic energy of the turbulent motion is about $3u_1^2$, both per unit mass, the turbulent energy of a fluid parcel entering a region of

† Kinematic stresses are used, i.e. the mechanical stresses divided by the fluid density.

changed rate-of-strain cannot change appreciably in a time much less than $3kz/u_1$. In this time, the mean flow will move the parcel a distance

$$x = \frac{3kz}{u_1} U(z) = 3z \log \frac{z}{z_1}. \tag{2.2}$$†

Since the influence of the change of surface begins near $x = 0$, turbulent energy and Reynolds stress are almost unchanged along streamlines well above the critical surface defined by equation (2.2). It follows that the stress gradient has the same value as far upstream, i.e. very nearly zero, so flow acceleration is negligible and mean velocity remains constant along streamlines far above the critical surface. In short, the only modification caused by the change of surface is a vertical displacement of the streamlines.

The same argument shows that the turbulent energy and Reynolds stress of a fluid parcel can adjust to a changed rate of shear if the parcel moves a distance large compared with $3z \log z/z_0$ in substantially constant shear (z_0 is the local surface roughness). So the Reynolds stress is determined by local rates of shear at heights small compared with the height of the critical surface unless the roughness length varies rapidly. In other words, there is an equilibrium layer (Townsend 1961) with a distribution of mean velocity,

$$U = \frac{\tau_0^{\frac{1}{2}}}{k} \log \frac{z}{z_0}, \tag{2.3}$$

where τ_0 is the local surface stress. The distribution (2.3) sets the inner boundary condition for the modified flow. With the outer boundary condition of simple displacement of the streamlines, the test for the possibility of self-preserving flow can be applied.

3. Streamline displacement and the velocity distribution

Far above the critical surface, each streamline is displaced outwards by the displacement thickness $\delta_1(x)$, which depends on the net velocity changes along streamlines in the region of accelerated flow and which causes a change in the velocity at constant height of $-u_1\delta_1/(kz)$ (figure 1). It is useful to distinguish the change in the velocity profile by flow acceleration from the change due to streamline displacement. Assuming $\delta(z)$, the streamline displacement at height z, to be small compared with z, we write the mean velocity as

$$U(z) = U_1(z) - u_1\delta(z)/(kz) + V(z), \tag{3.1}$$

where $u_1/(kz)$ is the velocity gradient in the basic flow from equation (2.1). The component $V(z)$ is determined by the flow accelerations and is expected to become very small above the critical surface. The streamline displacement is given by the condition of incompressibility

$$\int_0^z U_1(z')\,dz' = \int_0^{z+\delta} U(z')\,dz',$$

† In atmospheric boundary layers, the kinetic energy may be considerably larger than $3u_1^2$, but the excess is associated with swirling motions which contribute little to the Reynolds stress.

or, to the approximation that δ/z is small, by

$$\int_0^z (U_1 - U)\,dz' = \delta U(z). \tag{3.2}$$

Substituting the form (3.1)

$$\int_0^z \left(V - \frac{u_1\delta}{kz'}\right) dz' = -\delta(z)\left\{\frac{u_1}{k}\log\frac{z}{z_1} - \frac{u_1\delta}{kz} + V\right\}. \tag{3.3}$$

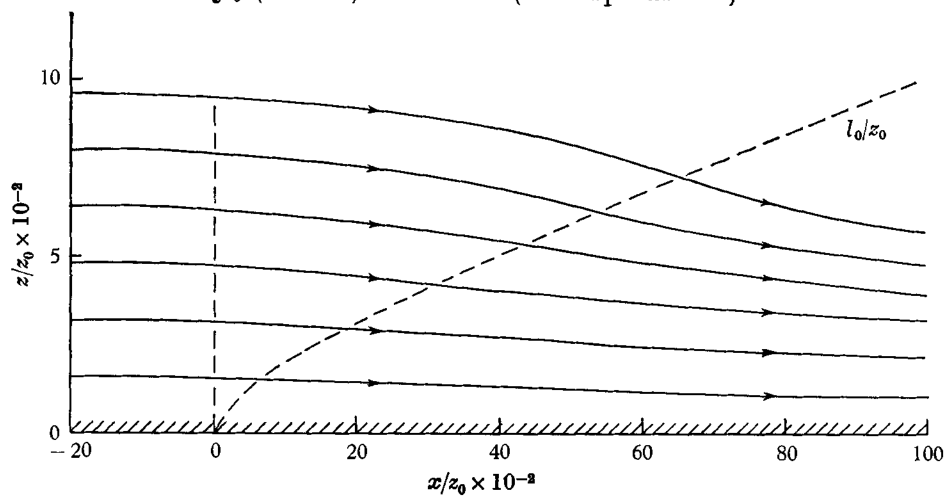

FIGURE 1. Vertical section of the flow showing streamlines of the mean flow in relation to the surface $z = l_0(x)$. The displacement of the streamlines would be appropriate to a value of M near 10.

Defining
$$C_0 = \lim_{z\to\infty}\left[\int_0^z \frac{\delta(z')}{\delta_1}\frac{dz'}{z'} - \log\frac{z}{l_0}\right],$$

where l_0 is the thickness of the accelerated region (to be defined later),

$$\int_0^\infty V\,dz = -\frac{u_1\delta_1}{k}(\log l_0/z_1 - C_0). \tag{3.4}$$

It is implied that l_0 has been chosen so that $|\log z/l_0|$ is much smaller than $\log l_0/z_1$ over most of the accelerated region and so that C_0 is of order one. By definition of the streamline displacement,

$$V = -U_1\frac{\partial\delta}{\partial z} = -\frac{u_1}{k}\log\frac{z}{z_1}\frac{\partial\delta}{\partial z}.$$

Integrating by parts in equation (3.3) and using this,

$$\int_0^z V\left(1 - \frac{\log z'/l_0}{\log z'/z_1}\right) dz' = -\frac{u_1\delta}{k}(\log l_0/z_1 - 1 + kV/u_1).$$

It follows that

$$\frac{\delta}{\delta_1} = \frac{\log l_0/z_1 - 1}{\log l_0/z_1 - 1 + kV/u_1}\,\frac{\int_0^z V\{1 + (\log z'/l_0)/(\log l_0/z_1)\}^{-1}\,dz'}{\int_0^\infty V\{1 + (\log z'/l_0)/(\log l_0/z_1)\}^{-1}\,dz'}.$$

If kV/u_1 is of order one or smaller, to order $(\log l_0/z_1)^{-1}$

$$\frac{\delta}{\delta_1} = \int_0^z V(z')\,dz' \Big/ \int_0^\infty V(z)\,dz. \tag{3.5}$$

Again, exactly

$$C_0 = -\frac{\int_0^\infty V \log z/l_0 \{1 + (\log z/l_0)/(\log l_0/z_1)\}^{-1} dz}{\left\{1 - (\log l_0/z_1)^{-1} \int_0^\infty V \log z/l_0 \{1 + (\log z/l_0)/(\log l_0/z_1)\}^{-1} dz\right\} \int_0^\infty V\,dz},$$

but to order $(\log l_0/z_1)^{-1}$

$$C_0 = -\int_0^\infty V \log z/l_0\,dz \Big/ \int_0^\infty V\,dz. \tag{3.6}$$

For small values of z/l_0, V varies as $\log z/z_0$, and so δ/δ_0 is nearly $Vz \Big/ \int_0^\infty V\,dz$. The ratio of the velocity change by displacement to the change by acceleration is nearly

$$\frac{u_1 \delta}{kzV} = \frac{-1}{\log l_0/z_1 - C_0},$$

and the ratio δ/z does not exceed V/U_1.

The total momentum added to the basic flow in the fetch $0-x$ is

$$P_x = \int_0^x [u_1^2 - \tau_0(x')]\,dx' = \int_0^{Z+\delta} U^2(z')\,dz' - \int_0^Z U_1^2(z')\,dz'$$

$$= \int_0^Z (U^2 - U_1^2)\,dz + \delta_1 U_1^2(Z), \tag{3.7}$$

where Z is a height much greater than that of the critical surface. Substituting the form (3.1),

$$P_x = \int_0^\infty \left(V - \frac{u_1 \delta}{kz}\right)^2 dz + \frac{2u_1}{k} \int_0^\infty V \log \frac{z}{z_1}\,dz$$

$$+ \frac{\delta_1 u_1^2}{k^2} \lim_{Z\to\infty} \left[\log^2 \frac{Z}{z_1} - 2\int_0^Z \frac{\delta}{\delta_1} \log \frac{z}{z_1} \frac{dz}{z}\right]. \tag{3.8}$$

Using the values of δ and C_0 given by equations (3.5) and (3.6), the last term reduces to

$$\frac{\delta_1 u_1^2}{k^2} \int_0^\infty V \log^2 \frac{z}{z_1}\,dz \Big/ \int_0^\infty V\,dz,$$

and the complete expression for the momentum flux becomes

$$P_x = \int_0^\infty \left(V - \frac{u_1 \delta}{kz}\right)^2 dz + \frac{2u_1}{k} \int_0^\infty V \log \frac{z}{z_1}\,dz$$

$$- \frac{u_1}{k} \left(\log \frac{l_0}{z_1} - C_0\right)^{-1} \int_0^\infty V \log^2 \frac{z}{z_1}\,dz. \tag{3.9}$$

Putting $\log z/z_1 = \log z/l_0 + \log l_0/z_1$ and remembering that $\log l_0/z_1 \gg |\log z/l_0|$ over nearly the whole range of integration if $\log l_0/z_1$ is large, the leading terms are found to be

$$P_x = \frac{u_1}{k} (\log l_0/z_1 - C_0) \int_0^\infty V\,dz + \int_0^\infty \left(V - \frac{u_1 \delta}{kz}\right)^2 dz$$

$$+ O\left(V \frac{u_1 l_0}{k^2} \left(\log \frac{l_0}{z_1}\right)^{-1}\right). \tag{3.10}$$

If $V \ll (u_1/k)\log l_0/z_1$, only the first term need be retained, the remaining terms being smaller by a factor of $(\log l_0/z_1)^{-2}$.

When transport of heat or any passive scalar quantity is considered, a similar distinction between changes of profile caused by displacement and by lateral transport is useful. With the basic temperature distribution

$$T_1(z) = -\frac{\theta_1}{k}\log\frac{z}{z_1},$$

the increase in temperature due to streamline displacement is $\theta_1\delta/(kz)$, and we write the mean temperature as

$$T(z) = T_1(z) + \theta_1\delta/(kz) + S(z). \tag{3.11}$$

The condition of conservation of heat is expressed by

$$Q_x = \int_0^x (Q_0(x') - Q_1)\,dx' = \int_0^Z (U(z')\,T(z') - U_1(z')\,T_1(z'))\,dz' + U_1(Z)\,T_1(Z)\,\delta_1, \tag{3.12}$$

where Q_x is the additional heat communicated to the fluid as a result of the change in heat flux from the surface from the upstream value Q_1 to the value $Q_0(x)$ at position $x > 0$ (the fluxes are thermometric).

Substituting the form (3.11),

$$Q_x = \int_0^\infty \left(S + \frac{\theta_1\delta}{kz}\right)\left(V - \frac{u_1\delta}{kz}\right)dz + \frac{u_1}{k}\int_0^Z \left(S + \frac{\theta_1\delta}{kz}\right)\log\frac{z}{z_1}\,dz - \frac{\theta_1}{k}\int_0^Z \left(V - \frac{u_1\delta}{kz}\right)\log\frac{z}{z_1}\,dz - \frac{u_1\theta_1\delta_1}{k^2}\log^2\frac{Z}{z_1}. \tag{3.13}$$

Using equations (3.4) and (3.5) for δ_1 and δ/δ_1, we find

$$Q_x = \int_0^\infty \left(S + \frac{\theta_1\delta}{kz}\right)\left(V - \frac{u_1\delta}{kz}\right)dz + \frac{u_1}{k}\int_0^\infty S\log\frac{z}{z_1}\,dz - \frac{\theta_1}{k}\int_0^\infty V\log\frac{z}{z_1}\,dz + \frac{\theta_1}{k}\left(\log\frac{l_0}{z_1} - C_0\right)^{-1}\int_0^\infty V\log^2\frac{z}{z_1}\,dz. \tag{3.14}$$

For large values of $\log l_0/z_1$, the leading terms are

$$Q_x = \frac{u_1}{k}\left(\log\frac{l_0}{z_1}\int_0^\infty S\,dz + \int_0^\infty S\log\frac{z}{l_0}\,dz\right) + \int_0^\infty \left(S + \frac{\theta_1\delta}{kz}\right)\left(V - \frac{u_1\delta}{kz}\right)dz + O\left\{\frac{\theta_1 V l_0}{k}\left(\log\frac{l_0}{z_1}\right)^{-1}\right\}. \tag{3.15}$$

4. First kind of velocity perturbation (change of roughness)

The simplest kind of self-preserving velocity perturbation is one with an acceleration component of the form

$$V = U - U_1 + \frac{u_1\delta}{kz} = \frac{u_0}{k}f(z/l_0), \tag{4.1}$$

where u_0 is the scale of the change in mean velocity, and l_0 is the length scale of the disturbed region and is comparable in height with the critical surface. Both scales are functions of x alone. The inner boundary condition is that the velocity distribution assumes the form

$$U = \frac{\tau_0^{\frac{1}{2}}}{k}\log\frac{z}{z_0}$$

for small values of $\eta = z/l_0$. For this to be possible, it is necessary that:

$$\text{(i)}\quad f(z/l_0) = \log z/z_0 + C \quad \text{for small} \quad z/l_0, \tag{4.2}$$

$$\text{(ii)}\qquad \tau_0^{\frac{1}{2}} = u_1 + u_0\{1 + (\log l_0/z_0 - M - C_0)^{-1}\}, \tag{4.3}$$

and

$$\text{(iii)}\quad \frac{u_0}{u_1} = -\frac{M}{\log l_0/z_0 - C + 1}, \tag{4.4}$$

where $M = \log z_1/z_0$ measures the change in roughness. Equations (3.4) and (3.5) have been used to evaluate the displacement term for the distribution function of (4.2), and terms of order $(\log l_0/z_0)^{-2}$ have been omitted in the condition (4.4).

The self-preserving form for the stress distribution is

$$\tau = u_1^2 + \tau_s F(z/l_0), \tag{4.5}$$

and necessarily $\tau_0 - u_1^2 = \tau_s$ or a multiple of it. Consistency with the condition (4.3) is possible if $|u_0/u_1|$ is small, when

$$\tau_s = 2u_0 u_1. \tag{4.6}$$

Having satisfied the boundary conditions, the distributions are now substituted in the Reynolds equation for the mean velocity

$$U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial z} = \frac{\partial \tau}{\partial z}$$

to obtain

$$U_1\left(\frac{du_0}{dx}f - \frac{u_0}{l_0}\frac{dl_0}{dx}\eta f'\right) + \frac{dU_1}{dz}\left(u_0\frac{dl_0}{dx}\eta f - \frac{d(u_0 l_0)}{dx}\int_0 f d\eta\right) = \frac{2ku_0u_1}{l_0}F'. \tag{4.7}$$

It has been assumed that $|U - U_1| \ll U_1$, and terms involving the displacement component $u_1\delta/(kz)$ have been omitted since their ratio to terms involving V is of order $(\log l_0/z_0)^{-1}$ at most. After putting in the values of U_1 and dU_1/dz, the equation is seen to be satisfied by the functions $f(\eta)$ and $F(\eta)$ if the non-dimensional coefficients

$$\frac{l_0}{u_0}\frac{du_0}{dx}\log\frac{l_0}{z_0},\quad \frac{dl_0}{dx}\log\frac{l_0}{z_0},\quad \frac{dl_0}{dx},\quad \frac{l_0}{u_0}\frac{du_0}{dx}$$

are either constant or negligibly small. For the large values of $\log l_0/z_0$ necessary to make $|u_0/u_1|$ small and for not too small η, the second coefficient is much the largest. For small values of η, all the terms are comparable but the flow is self-preserving since it is part of an equilibrium layer which adjusts itself rapidly to the surface stress. A similar situation occurs in the simple self-preserving boundary layer (Townsend 1956). To the approximation of large $\log l_0/z_0$, the equation is satisfied by the self-preserving distributions of velocity and stress if

$$\frac{dl_0}{dx}\log\frac{l_0}{z_0} = 2k^2, \tag{4.8}$$

the particular value of the constant being chosen for reasons that will become clear later. From the relation (4.4) between u_0 and l_0, the first coefficient is

$$\frac{-2k^2}{\log l_0/z_0 - C - 1} + \frac{l_0}{M}\frac{dM}{dl_0} \tag{4.9}$$

and is negligible unless $(x/M)\,dM/dx$ becomes appreciable. The self-preserving form of the equation of motion valid for large $\log l_0/z_0$ is

$$-\eta f' = F'. \tag{4.10}$$

Similar arguments show that the Reynolds equation for the kinetic energy of the turbulent motion is satisfied by the self-preserving distributions

$$\left.\begin{aligned} \tfrac{1}{2}\overline{q^2} &= \tfrac{1}{2}\overline{q_1^2} + u_0 u_1 Q(\eta), \\ \tfrac{1}{2}\overline{q^2 w} + \overline{pw} &= \tfrac{1}{2}\overline{q_1^2 w_1} + \overline{p_1 w_1} + u_1^2 u_0 D(\eta), \\ \epsilon &= \epsilon_1 + \frac{u_0 u_1^2}{l_0} E(\eta), \end{aligned}\right\} \tag{4.11}$$

where the suffix 1 indicates conditions for the basic flow and the usual notation is used for the turbulent fluctuations. To sum up, self-preserving flow is dynamically possible if

(i) the velocity-defect ratio $|U - U_1|/U_1$ is small, except for small z/l_0,

(ii) the variation of the roughness parameter M is slow, i.e.

$$d(\log |M|)/d(\log x) \ll 1$$

(the conditions are satisfied for constant $d(\log|M|)/d\log x$, but the corresponding distributions of roughness length are rather special and unlikely to occur),

(iii) $\log l_0/z_0$ is large.

The development of the flow is then described by equations (4.4) and (4.8). Notice that the equation for the layer thickness integrates to

$$l_0(\log l_0/z_0 - 1) = 2k^2 x, \tag{4.12}$$

showing that l_0 is quite close to the height of the critical surface (equation 2.2). A full solution of the mean-flow problem would include the explicit form of the distribution function $f(\eta)$, but the greater part of its variation is in the region of small z/l_0 where it is necessarily logarithmic as given by equation (4.2). So far as the change in surface stress is concerned, the function $f(\eta)$ influences its value only through the constant C which is of order one. For large $\log l_0/z_0$, C has a negligible influence on the stress.

It may appear from equation (4.4) that the surface friction approaches asymptotically its original value, irrespective of the change in roughness, but the theory depends on the outer flow being unmodified by the change of roughness. When l_0, calculated from equation (4.12), is comparable with the total thickness of the boundary layer, the theory is no longer applicable, but it is interesting that the surface stress given by the theory is then very near the equilibrium value for a boundary layer of the same thickness over a surface of the new roughness (Panofsky & Townsend 1964).

An explicit form for the distribution functions can be obtained only by making some assumptions about the interaction between the velocity field and the turbulent motion. The assumption of self-preserving flow simplifies the discussion of the interaction in terms of the energy balance, and a second equation relating the stress and velocity distribution functions can be obtained by making plausible assumptions about the lateral diffusion of turbulent energy and structural similarity (see appendix). The derivation of the distribution functions in this way is rather more difficult than the importance of the problem justifies, and it is simpler to use the 'mixing-length' transfer relation

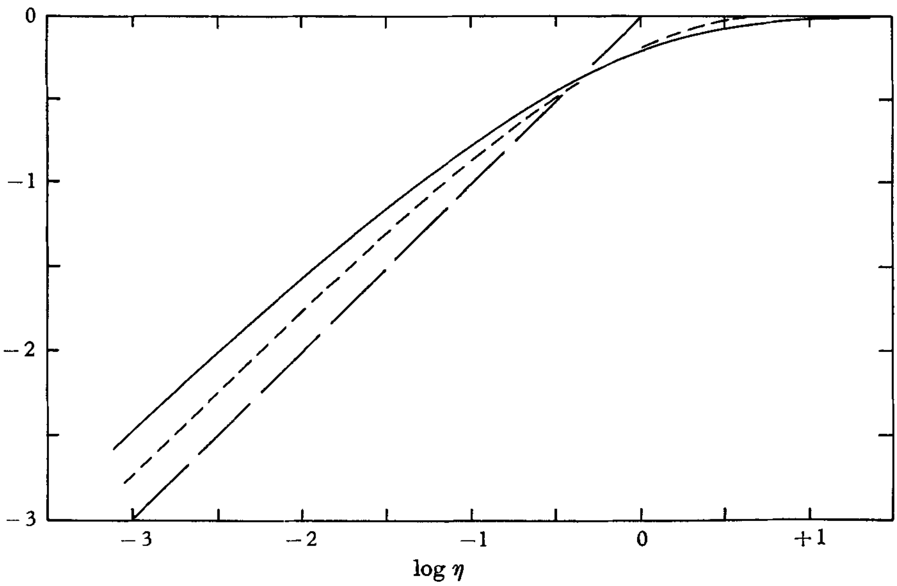

FIGURE 2. Comparison of distribution functions describing the velocity changes induced by a change of surface roughness. ———, $-\int_\eta^\infty e^{-x}/x\,dx$ (mixing length); – – –, $\log\frac{1}{2}\eta+(1+\frac{1}{2}\eta)$ (Panofsky & Townsend); — —, $\log\eta$ (Elliott).

$$\frac{\partial U}{\partial z} = \frac{\tau^{\frac{1}{2}}}{kz}. \tag{4.13}$$

In terms of the self-preserving functions, it is

$$f' = \eta^{-1}F. \tag{4.14}$$

Substituting in equation (4.10) and using the boundary condition $F(0) = 1$,

$$F(\eta) = e^{-\eta}, \tag{4.15}$$

and

$$f(\eta) = -\int_\eta^\infty \frac{e^{-x}}{x}dx = +Ei(-\eta). \tag{4.16}$$

The function $Ei(-x)$ approaches $+\log x + \gamma$ for small x (where $\gamma = 0{\cdot}577$ is Euler's constant), and

$$-Ei(-\eta) \sim e^{-\eta}\left(\frac{1}{\eta} - \frac{1}{\eta^2} + \frac{2!}{\eta^3} - \dots\right)$$

for large η. The constant $C = \gamma$, and

$$\frac{u_0}{u_1} = -\frac{M}{\log l_0/z_0 - \gamma + 1}. \tag{4.17}$$

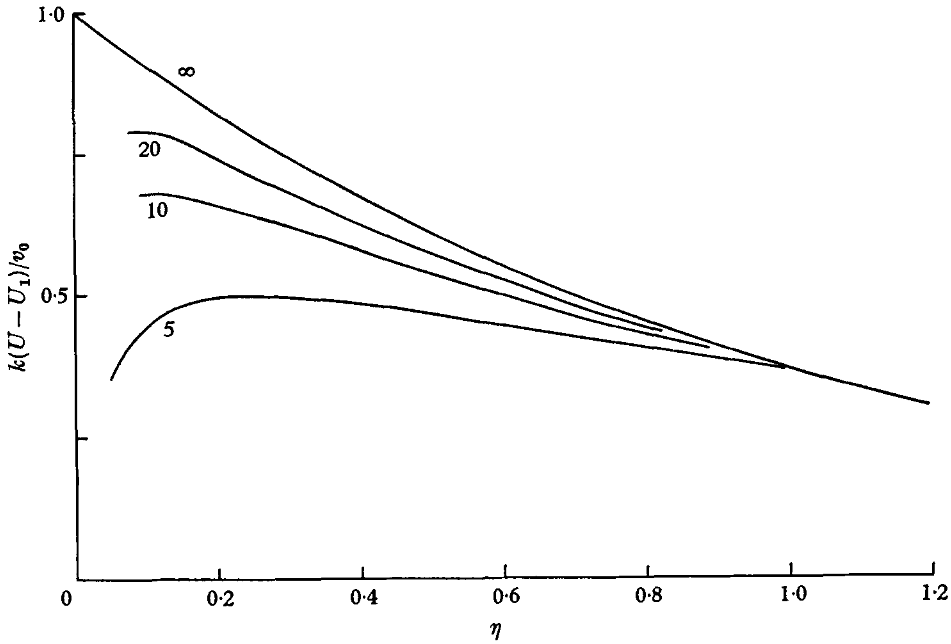

Figure 3. The change of velocity downstream of a line roughness, showing the relative magnitudes of the two components for various values of $\log l_0/z_0$. (The numbers on the curves are values of $\log l_0/z_0$.)

The solutions by Elliott (1958) and by Panofsky & Townsend (1964) assume particular forms for the velocity distribution. Elliott assumes a logarithmic distribution

$$f(\eta) = \log \eta \quad (\eta \leqslant 1)$$

in the present notation, for which

$$\frac{u_0}{u_1} = -\frac{M}{\log l_0/z_0 + 1}.$$

Panofsky & Townsend assume a log-linear distribution,

$$f(\eta) = \log \tfrac{1}{2}\eta + (1 - \tfrac{1}{2}\eta),$$

for which

$$\frac{u_0}{u_1} = -\frac{M}{\log l_0/z_0 + \log_e 2}.$$

For large $\log l_0/z_0$, the various predictions of surface stress are very near each other, and figure 2 shows the differences in velocity profile.

5. Second kind of velocity perturbation (line-roughness)

The first kind of self-preserving flow is possible only with a step change in the surface roughness since the condition (4.4) demands no flow disturbance if the roughness regains its upstream value. If localized roughness exists near the line $x = 0$, perhaps in the form of a fence, another kind of self-preserving flow disturbance is possible, but it contains two components of disturbance. Qualitatively, the components are the wake of the localized roughness which is dominant at heights of order l_0, and a flow change associated with the lessened surface stress arising from sheltering. We put

$$U - U_1 = -\frac{u_1\delta}{kz} + \frac{u_0}{k} f(z/l_0) + \frac{v_0}{k} g(z/l_0), \tag{5.1}$$

where the first term is the streamline-displacement term, the second the change due to change of surface stress, and the third the wake component. As before, u_0, v_0 and l_0 are functions of x, and correct behaviour in the equilibrium layer near the surface and far above the critical surface is obtained if

(i) $f(\eta) = \log \eta + C_1$ for small η, $g(0) = 1$

and $f(\eta) \to 0$, $g(\eta) \to 0$ for large η,

$$\text{(ii)} \quad \tau_0^{\frac{1}{2}} = u_1 + u_0\{1 + (\log l_0/z_0 - C_0)^{-1}\}, \tag{5.2}$$

and

$$\text{(iii)} \quad u_0(\log l_0/z_0 - C_1) = v_0. \tag{5.3}$$

It follows from the last condition that the wake component is much larger than the surface component except very near the surface. The self-preserving distribution of stress is

$$\tau - u_1^2 = 2u_0u_1F(z/l_0) + 2v_0u_1G(z/l_0), \tag{5.4}$$

where $F(0) = 1$, $H(0) = 0$ and both become small for large values of z/l_0. Substituting the forms in the equation for the mean velocity, it can be confirmed that self-preserving flow is dynamically possible under much the same conditions as before, i.e. small velocity defect as a fraction of the local velocity and large $\log l_0/z_0$. To the approximation in use, the equation of mean flow is

$$\log\frac{l_0}{z_0}\left\{\frac{l_0}{v_0}\frac{dv_0}{dx}g - \eta g'\frac{dl_0}{dx}\right\} = 2k^2G' \tag{5.5}$$

for not too small values of η, and it is of self-preserving form if

$$\frac{dl_0}{dx}\log\frac{l_0}{z_0} = 2k^2 \quad \text{and} \quad \frac{l_0}{v_0}\frac{dv_0}{dx}\log\frac{l_0}{z_0} = \text{const.} \tag{5.6}$$

Although the flow for small values of z/l_0 is determined by the surface stress and so by the velocity scale u_0, the contribution of the surface function to the total velocity change is very small. To find the variation of the velocity scales, consider the rate of change of momentum flux in the whole flow. From equation (3.10), the flux additional to that of the basic flow is

$$P_x = \frac{u_1v_0l_0}{k^2} I_0 \log\frac{l_0}{z_0}$$

to the approximation of self-preserving development, where

$$I_0 = \int_0^\infty g(\eta)\, d\eta.$$

The condition of conservation of total momentum is then

$$I_0 \frac{d}{dx}\left(v_0 l_0 \log \frac{l_0}{z_0}\right) = -2k^2 u_0. \tag{5.7}$$

Using the equation for l_0 (5.6) and the relation (5.3) between u_0 and v_0, we find that

$$v_0 \propto l_0^{-1}(\log l_0/z_0)^{-1-1/I_0}. \tag{5.8}$$

To the approximation in use, then,

$$\frac{l_0}{v_0}\frac{dv_0}{dx}\log\frac{l_0}{z_0} = -2k^2,$$

and the equation of motion takes the non-dimensional form,

$$g + \eta g' = -G', \tag{5.9}$$

i.e.

$$\eta g = -G,$$

since $G(0) = 0$.

Equation (5.8) indicates a slow decrease of the difference-momentum flux with distance from the line roughness, caused by the change in surface friction. In fact

$$P_x \propto (\log l_0/z_0)^{-1/I_0}, \tag{5.10}$$

the constant of proportionality depending on the nature of the line-roughness. For a fence of height h, its order of magnitude could be found by equating the drag on the fence to the defect of momentum flux when the height of the modified layer equalled the fence height. If the drag on the fence is

$$C_d \frac{u_1^2 h}{k^2}\log^2\frac{h}{z_0} = C_d h U_1^2(h),$$

the magnitudes of the scales are given (by (5.6) and (5.8)) as

$$\left.\begin{aligned} 2k^2 x &= l_0(\log l_0/z_0 - 1) - h(\log h/z_0 - 1), \\ I_0 v_0 l_0 (\log l_0/z_0)^{1+1/I_0} &= C_d u_1 h (\log h/z_0)^{2+1/I_0}. \end{aligned}\right\} \tag{5.11}$$

As before, the mixing-length relation is used to obtain a velocity profile. Not too close to the surface it gives

$$g' = \eta^{-1} G,$$

and substitution in equation (5.9) leads to

$$g + g' = 0, \tag{5.12}$$

with solution

$$g(\eta) = e^{-\eta}. \tag{5.13}$$

The distribution function for stress is

$$G(\eta) = -\eta e^{-\eta}. \tag{5.14}$$

The surface component of the flow can be found only by considering aspects of the flow that are not self-preserving. As an illustration, put

$$\begin{aligned} f(\eta) &= \log \eta \quad \text{for} \quad \eta < 1, \\ &= 0 \quad \text{for} \quad \eta > 1. \end{aligned}$$

Then the complete distribution is

$$\left.\begin{aligned} U - U_1 &= -\frac{v_0}{k}\left(e^{-\eta} + \frac{\log \eta}{\log l_0/z_0}\right) \quad \text{for} \quad \eta < 1, \\ &= -(v_0/k)\, e^{-\eta} \quad \text{for} \quad \eta > 1. \end{aligned}\right\} \tag{5.15}$$

The velocity distribution is shown in figure 3 for several large values of $\log l_0/z_0$. The diminishing importance of the surface component for large $\log l_0/z_0$ is clearly shown.

6. Disturbance of a boundary layer on a flat plate

A distinct and interesting example of a self-preserving perturbation of a basic flow concerns a boundary layer on a flat plate which is approaching the asymptotic self-preserving form of development with velocity and Reynolds stress distributions of the forms

$$\begin{aligned} U_1(z) &= U_0 - (u_1/k) f_1(z/\delta_0), \\ \tau_1(z) &= u_1^2 F_1(z/\delta_0), \end{aligned}$$

where U_0 is the velocity of the free stream, and u_1 and δ_0 are scales of velocity and length. It is well known that self-preserving development of this kind is dynamically possible at large Reynolds numbers if u_1 is the friction velocity and if $\delta_0 = z_0 \exp(kU_0/u_1)$ and varies with distance from the flow origin as specified by

$$\log \frac{\delta_0}{z_0} \frac{d\delta_0}{dx} = \frac{k^2}{I_1}, \quad \text{where} \quad I_1 = \int_0^\infty f_1(\eta)\, d\eta \tag{6.1}$$

(see, for example, Townsend 1956). In a particular layer, the distributions of velocity and stress may depart from the self-preserving forms but approach them asymptotically with increase of downstream distance. The causes of the departures may be found in details of the transition process, the presence of surface-roughness upstream or possibly earlier development in an adverse pressure gradient. At least in the later stages of development, the deviations from the basic self-preserving form are distributed over the whole thickness of the layer and not confined to a thin surface layer like the flows considered in the previous sections. Then, if we look for self-preserving perturbations of the basic flow, we must use the same length scale δ_0 and may postulate a perturbation of a form similar to that used in the previous section

$$\left.\begin{aligned} U - U_1 &= \frac{u_0}{k} f(z/\delta_0) + \frac{v_0}{k} g(z/\delta_0), \\ \tau - \tau_1 &= 2\, u_1 u_0 F(z/\delta_0) + 2 u_1 v_0 G(z/\delta_0). \end{aligned}\right\} \tag{6.2}$$

The presence of perturbations at the outer limit of the flow makes unnecessary the inclusion of a displacement component of the change of velocity. The distribution functions satisfy the same boundary conditions as before and so

$$u_0(\log \delta_0/z_0 - C_1) = v_0. \tag{6.3}$$

Substitution in the equation of mean flow confirms that the self-preserving development is possible for small perturbations and large values of $\log \delta_0/z_0$. The leading terms of the equation are

$$\frac{U_0}{u_1}\left(\frac{\delta_0}{v_0}\frac{dv_0}{dx}g - \frac{d\delta_0}{dx}\eta g'\right) = 2kG'. \tag{6.4}$$

Since $U_0 = (u_1/k)\log \delta_0/z_0$, constancy of the coefficient ratios is obtained if $v_0 \propto \delta_0^n$. To find the exponent, we use the condition of conservation of total momentum

$$\frac{d}{dx}\left[I_0\frac{v_0 U_0 \delta_0}{k}\right] = -2u_1 u_0, \tag{6.5}$$

where
$$I_0 = \int_0^\infty g(\eta)\,d\eta.$$

Using the condition (6.3),

$$v_0 \propto \delta_0^{-1}(\log \delta_0/z_0)^{-1-2I_1/I_0}, \tag{6.6}$$

and the non-dimensional form of the equation of mean flow is

$$-g - \eta g' = 2I_1 G'. \tag{6.7}$$

For a simple boundary layer, the relation between Reynolds stress and velocity gradient in the outer part of the flow is adequately described by a coefficient of eddy viscosity ν_T related to the velocity distribution by

$$\nu_T = R_s^{-1}\int_0^\infty (U_0 - U_1)\,dz$$

(Townsend 1961). To the approximation of small disturbance, the change in Reynolds stress is

$$\tau - \tau_1 = \frac{u_1\delta_0 I_1}{kR_s}\frac{\partial(U-U_1)}{\partial z}, \tag{6.8}$$

or, in non-dimensional form,

$$2k^2R_s G = I_1 g'. \tag{6.9}$$

Combining the eddy-viscosity relation with the equation of mean flow leads to the equation for the velocity-distribution function,

$$-g - \eta g' = (I_1^2/k^2R_s)\,G''. \tag{6.10}$$

The appropriate solution is

$$g(\eta) = e^{-\frac{1}{2}R^2\eta^2}, \tag{6.11}$$

where $R = kR_s^{\frac{1}{2}}/I_1$. With the same eddy viscosity, the basic velocity distribution is

$$U_0 - U_1 = C'R\int_\eta^\infty e^{-\frac{1}{2}R^2x^2}\,dx, \tag{6.12}$$

with much the same asymptotic behaviour as the velocity perturbation.

The magnitudes of the velocity and stress changes may be compared using the eddy-viscosity assumption. In the outer flow, the maximum change is v_0/k or, expressed as a fraction of the free stream velocity, $v_0/(u_1 \log \delta_0/z_0)$. The change of stress is

$$\tau - \tau_1 = u_1 v_0 (kR_s^{\frac{1}{2}})^{-1} R\eta\, e^{-\frac{1}{2}R^2\eta^2}. \tag{6.13}$$

The maximum change expressed as a fraction of the surface stress is

$$(v_0/u_1)\, e^{-\frac{1}{2}}/(kR_s^{\frac{1}{2}}).$$

From measured profiles, $kR_s^{\frac{1}{2}} = 3{\cdot}2$ and $I_1 = 0{\cdot}55$. For a fairly thick boundary layer in a wind tunnel, $\log \delta_0/z_0$ is about ten, and the fractional change in stress is about three times the fractional change in velocity. In practice, the stress changes should be much more evident than the velocity changes since they reach a maximum near the middle of the layer while the velocity changes have much the same form as the basic profile.

7. Self-preserving distributions of temperature

The Eulerian concept of self-preserving development can be and has been extended to describe the turbulent diffusion of a convected scalar such as temperature, and it is equivalent to the concept of Lagrangian similarity developed by Batchelor (1957), by Ellison (1959) and by Cermak (1963). If the diffusion takes place in a flow undergoing modification through changes of surface conditions, the Eulerian method has the advantages of greater flexibility and of permitting tests of consistency with the equations of motion and of conservation of the diffused quantity. For convenience, consider diffusion of heat from surface sources in a deep boundary layer that passes from an upstream region of uniform surface roughness and heat-flux to a region of different, and possibly varying, surface roughness and surface flux. The heat flux is supposed so small that the consequent density variations are without effect on the motion, which is undergoing self-preserving development of the first kind described by the theory of §4. For simplicity, it is assumed that the fluid temperature extrapolated to the roughness height equals the physical temperature of the surface and that the eddy-transport coefficients for momentum and heat are equal *in equilibrium flows*, i.e.

$$T = T_g - \frac{Q_0}{k\tau_0^{\frac{1}{2}}} \log \frac{z}{z_0}, \tag{7.1}$$

where T_g is the local ground temperature, and Q_0 is the local thermometric flux from the surface. Neither of the assumptions is essential and modifications for departures are not difficult.

The argument is very similar to that used for the changes in flow velocity. For negative x, the whole flow is an equilibrium layer and, taking the upstream ground temperature as zero, the temperature distribution is

$$T_1 = -\frac{Q_1}{ku_1} \log \frac{z}{z_1}. \tag{7.2}$$

For positive x, the temperature distribution is written as

$$T - T_1 = -\frac{\theta_0}{k} \phi\left(\frac{z}{l_0}\right) + \frac{Q_1 \delta}{ku_1 z}, \tag{7.3}$$

where θ_0 is a scale of temperature dependent on x, l_0 is the length-scale of the modified flow given by equation (4.12), and the term $Q_1\delta/(ku_1 z)$ represents the change in temperature by streamline displacement. The inner boundary condition is that the temperature distribution has the form (7.1). Using the relations (3.4) and (3.5) for δ, the inner condition can be satisfied if

$$\left.\begin{aligned}\phi(\eta) &= \log\eta + C_2 \quad \text{for small } \eta,\\ \theta_0 &= \frac{Q_0-Q_1}{\tau_0^{\frac{1}{2}}} - \frac{Q_1}{u_1}\left(\frac{\tau_0^{\frac{1}{2}}-u_1}{\tau_0^{\frac{1}{2}}} + \frac{u_0}{u_1(\log l_0/z_1 - C_0)}\right),\\ ku_1T_g - MQ_1 &= (\log l_0/z_0 - C_2)\left\{(Q_0-Q_1)\frac{u_1}{\tau_0^{\frac{1}{2}}} - Q_1\frac{\tau_0^{\frac{1}{2}}-u_1}{\tau_0^{\frac{1}{2}}}\right\} + \frac{Q_1u_0(C_2-C+1)}{u_1(\log l_0/z_1 - C_0)}.\end{aligned}\right\} \tag{7.4}$$

Inspection of the last two conditions suggests that simple behaviour is likely (i) if $Q_1 = 0$ with Q_0 possibly a function of position, or (ii) if $Q_1 = Q_0$. The linearity of the heat equation means that the temperature distribution caused by the distribution of surface flux defined by Q_1 and Q_0 has the linear form

$$T = \frac{Q_0-Q_1}{ku_1}p(x,z) + \frac{Q_1}{ku_1}q(x,z), \tag{7.5}$$

where the functions p, q depend on the flow although p may depend also on the streamwise variation of Q_0/Q_1. It follows that combinations of the two special cases can be used to describe the temperature distributions for a wide variety of flux conditions.

For $Q_1 = 0$, the temperature scale is, very nearly,

$$\theta_0 = Q_0/(u_1+u_0), \tag{7.6}$$

and the surface temperature is given by

$$ku_1T_g = Q_0(\log l_0/z_0 + M - C_2). \tag{7.8}$$

A self-preserving distribution of heat flux is

$$\overline{w\theta} = q_0\Phi_1(z/l_0), \tag{7.9}$$

where q_0 must equal Q_0 since the flux is Q_0 at the surface and is zero for large values of z/l_0. Substituting the self-preserving forms for velocity and temperature in the heat equation,

$$U\frac{\partial T}{\partial x} + W\frac{\partial T}{\partial z} + \frac{\partial\overline{w\theta}}{\partial z} = 0,$$

and discarding terms of order $(\log l_0/z_0)^{-1}$, we obtain

$$\frac{u_1}{k^2}\log\frac{l_0}{z_0}\left\{-\frac{d\theta_0}{dx}\phi_1 + \frac{\theta_0}{l_0}\frac{dl_0}{dx}\eta\phi_1'\right\} + \frac{q_0}{l_0}\Phi_1' = 0. \tag{7.10}$$

Self-preserving development of the temperature field is possible with the length-scale given by equation (4.8) if the surface flux varies nearly as a power of the length scale. If $Q_0 \propto l_0^n$, the non-dimensional form of the heat equation is

$$-n\phi_1 + \eta\phi_1' + \tfrac{1}{2}\Phi_1' = 0. \tag{7.11}$$

To the approximation used, self-preserving development is also possible for $Q_0 \propto x^n$ with the same non-dimensional form of the heat equation and, if $Q_0(x)$ can be represented by a power series, the temperature distribution can be expressed as the superposition of self-preserving distributions. For this particular case, the change of roughness enters only through the lower boundary conditions, giving a shift of surface temperature and a small change of temperature scale.

Further confirmation of the possibility of self-preserving developments can be obtained by substituting forms for the distributions of $\overline{w^2\theta}$ and the rate of destruction of $\overline{w\theta}$ (analogous with the relations (4.11)) in the equation for the turbulent heat flux

$$\frac{\partial\overline{w\theta}}{\partial t}+U\frac{\partial\overline{w\theta}}{\partial x}+W\frac{\partial\overline{w\theta}}{\partial z}+\overline{w^2}\frac{\partial T}{\partial z}+\overline{w\theta}\frac{\partial U}{\partial z}+\frac{\partial}{\partial z}(\overline{w^2\theta})+\epsilon_{w\theta}=0.$$

The other special case $Q_1 = Q_0$, is trivial unless there is a change of roughness. The temperature scale is

$$\theta_0=\frac{MQ_0}{u_1(\log l_0/z_0-C+1)}\{1+(2+M)(\log l_0/z_1)^{-1}\}, \tag{7.12}$$

and the surface temperature is given by

$$ku_1T_g-2MQ_0=MQ_0\frac{M+2C-2C_2-1}{\log l_0/z_0}. \tag{7.13}$$

Substitution of the temperature distribution (7.3) and the flux distribution

$$\overline{w\theta}=Q_0+q_0\Phi_2(z/l_0), \tag{7.14}$$

leads to

$$\frac{u_1}{k^2}\left[(\phi_2+\eta\phi_2'\log l_0/z_0)\frac{\theta_0}{l_0}\frac{dl_0}{dx}\right]+\frac{q_0}{l_0}\Phi_2'=0. \tag{7.15}$$

For self-preserving development, $q_0 = \theta_0 u_1$ and $\log l_0/z_0$ must be large. The non-dimensional form is then

$$\eta\phi_2'+\tfrac{1}{2}\Phi_2'=0. \tag{7.16}$$

Notice that the surface temperature is nearly constant while the scales of temperature and flux vary as $(\log l_0/z_0)^{-1}$.

Explicit distributions may be obtained by assuming that the eddy diffusivity for heat is the same as for momentum and is given by the mixing-length relation, i.e. that

$$\overline{w\theta}=-k\tau^{\frac{1}{2}}z(\partial T/\partial z). \tag{7.17}$$

In terms of the distribution functions, the non-dimensional form is

$$q_0\Phi(\eta)=(u_0/u_1)Q_1G(\eta)+\theta_0u_1\eta\phi'(\eta) \tag{7.18}$$

for small perturbations. For $Q_1 = 0$, it becomes

$$\Phi_1=\eta\phi_1'$$

and can be substituted in the non-dimensional heat equation (7.11) to give

$$\tfrac{1}{2}\eta\phi_1''+(\eta+\tfrac{1}{2})\phi_1'-n\phi_1=0. \tag{7.19}$$

If Q_0 varies slowly ($n = d(\log Q_0)/d\log x \ll 1$), the distribution functions are

$$\left.\begin{aligned} \Phi_1(\eta) &= e^{-2\eta}, \\ \phi_1(\eta) &= -\int_\eta^\infty \frac{e^{-2\eta}}{\eta}\,d\eta = Ei(-2\eta), \\ &\approx \log\eta + \log 2 + \gamma \quad \text{for small } \eta, \\ &\sim e^{-2\eta}/2\eta \quad \text{for large } \eta. \end{aligned}\right\} \tag{7.20}$$

With this choice of transfer relation, $C_2 = \gamma + \log 2$. An interesting feature is that the temperature change falls off as $\exp(-2\eta)$ much more rapidly than the change of velocity induced by a change of roughness, although the eddy diffusivities for heat and momentum are assumed equal at all points.

For $Q_1 = Q_0$, the transfer relation is (using equation 4.15)

$$\Phi_2 = \eta\phi_2' - e^{-\eta},$$

and substitution in equation (7.16) leads to the distribution functions

$$\Phi_2(\eta) = -2(e^{-\eta} - e^{-2\eta}) \tag{7.21}$$

since $\Phi_2(0) = 0$, and

$$\left.\begin{aligned} \phi_2(\eta) &= \int_\eta^\infty \frac{e^{-\eta}}{\eta}\,d\eta - 2\int_\eta^\infty \frac{e^{-2\eta}}{\eta}\,d\eta, \\ &\approx \log\eta + 2\log 2 - \gamma \quad \text{for small } \eta, \\ &\sim \frac{e^{-\eta} - e^{-2\eta}}{\eta} \quad \text{for large } \eta. \end{aligned}\right\} \tag{7.22}$$

In this case, the approach to the original distribution is similar to that of the velocity field. As required by equation (7.16), $\int_0^\infty \phi_2(\eta)\,d\eta = 0$, and deviations from the original distribution occur with both signs.

The distribution functions obtained by assuming the mixing-length eddy diffusivity can be used to find the temperature distribution for a great variety of imposed distributions of surface flux, and, although the assumption is probably in error except for small values of z/l_0, they provide a reasonable approximation. Alternatively, experiments might be performed to determine empirically the functions $f(\eta)$ and $\phi(\eta)$ since the possibility of self-preserving development does not depend on the nature of the transfer process.

In practical problems, the surface flux may not be specified. If it is not, another surface condition is necessary, the most general being some functional relation between flux, surface temperature and position. Usually the consequent variation of surface flux in the stream direction will be sufficiently slow for the temperature distribution to be a linear combination of the distributions for the special cases considered above, using the third matching condition (7.4) to find the surface flux. For example, if the distribution of surface temperature is specified, the 'constant-flux' component of the distribution is known from the upstream value of the surface flux, and the 'constant-flux' distribution of ground

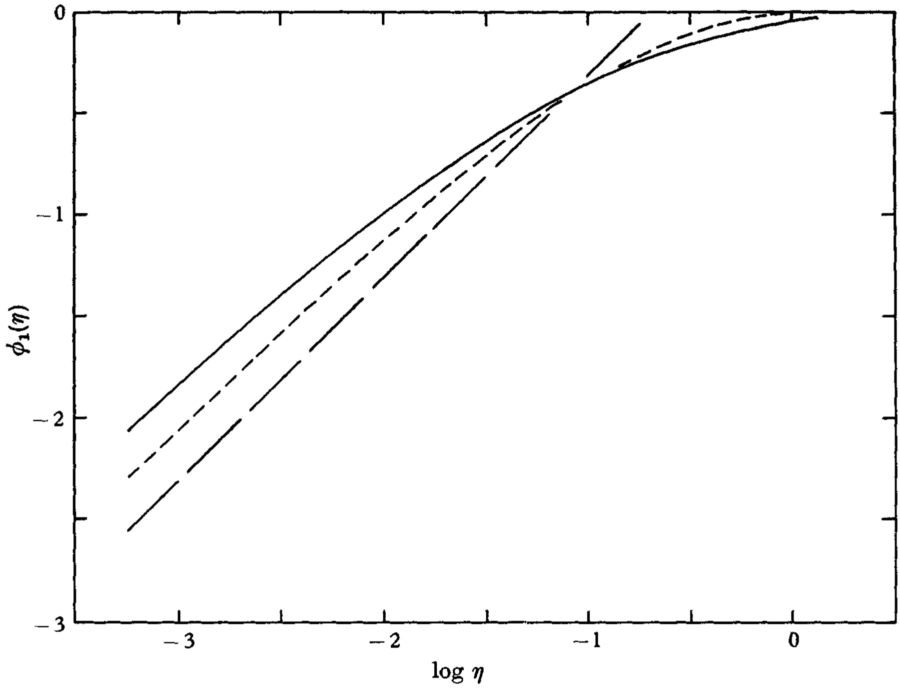

FIGURE 4. Comparison of distribution functions describing the temperature changes induced by a sudden change of surface heat-flux. ——, $-\int_\eta^\infty e^{-2x}/x\,dx$; ---, $\log \eta + (1-\eta)$; — —, $\log 2\eta$.

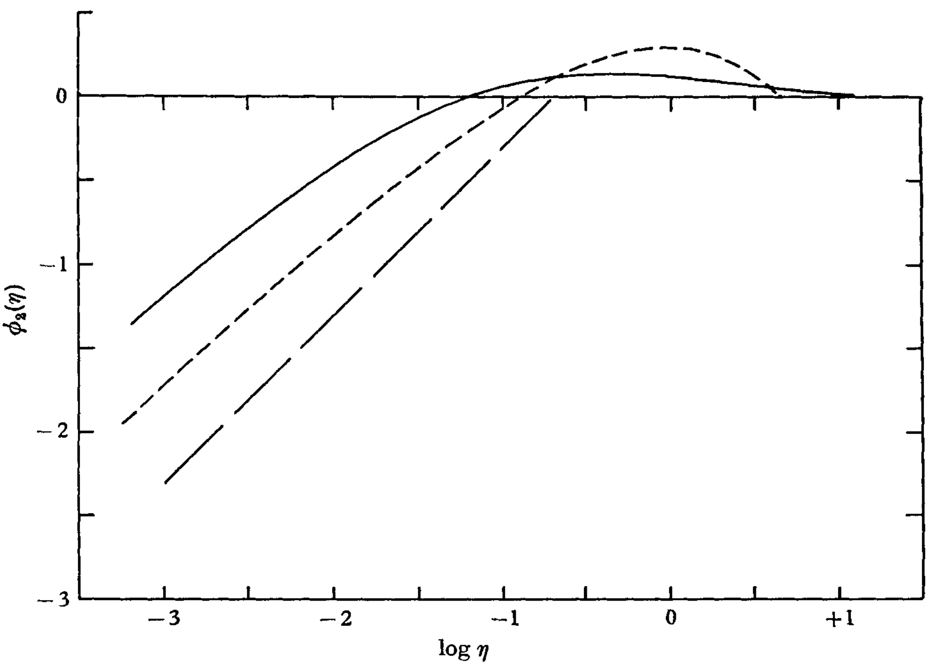

FIGURE 5. Comparison of distribution functions describing the temperature changes induced by a change of roughness without any change of surface heat-flux. ——, Mixing length; - - -, Panofsky & Townsend profile; — —, Elliott profile.

temperature is given by equation (7.13). The component induced by the change in flux has a temperature scale found by substituting for T_g in the equation

$$ku_1 T_g = (u_1+u_0)(\log l_0/z_0 + M - C_2)\,\theta_0 \tag{7.23}$$

(obtained from the conditions (7.6) and (7.7)) the difference between the specified surface temperature and the 'constant-flux' value. The requirements for self-preserving development of the two components will be met if the logarithmic rate of variation of this difference is small.

8. Diffusion from a line source of heat

The equivalent for heat transfer of flow modification by a line roughness is additional heat flux concentrated near the line $x = 0$, and self-preserving development of the temperature profile is possible of a similar kind, i.e. with

$$T = -\frac{\theta_w}{k}\psi(z/l_0) - \frac{\theta_0}{k}\phi_3(z/l_0), \tag{8.1}$$

where $\phi_3(0) = 1$, $\psi(\eta) = \log\eta + C_3$ for small values of η, and both functions approach zero for large η. The development exists in a pure form, i.e. uncomplicated by superimposed temperature fields of the kind considered in the previous section if both the upstream heat flux and the ground temperature are zero. Then the inner boundary condition is satisfied if

$$\left.\begin{aligned} &\theta_w(\log l_0/z_0 - C_3) = \theta_0, \\ \text{and}\quad &\theta_w = Q_0/(u_1+u_0), \end{aligned}\right\} \tag{8.2}$$

where Q_0 is the surface flux. The condition of overall conservation of heat leads to

$$\frac{d}{dx}\left(\frac{u_1\theta_0 l_0 I_3}{k^2}\log l_0/z_0\right) = -Q_0, \tag{8.3}$$

where

$$I_3 = \int_0^\infty \phi_3(\eta)\,d\eta,$$

showing that

$$\theta_0 l_0 \log l_0/z_0 \propto (\log l_0/z_0)^{-1/2I_3}. \tag{8.4}$$

Substitution in the heat equation then leads to the non-dimensional form

$$\phi_3 + \eta\phi_3' + \tfrac{1}{2}\Phi_3' = 0,$$

so that

$$\eta\phi_3 + \tfrac{1}{2}\Phi_3 = 0, \tag{8.5}$$

where the flux distribution is

$$\overline{w\theta} = Q_0\Psi(z/l_0) + q_0\,\Phi_3(z/l_0), \tag{8.6}$$

where $\Psi(0) = 1$, $\Phi_3(0) = 0$. As for the line roughness, the contribution of the wall component with scale θ_w is small over most of the modified layer, and the temperature at small, but not too small, heights is very nearly θ_0/k if $\log l_0/z_0$ is large. Approximately,

$$\theta_0 \propto x^{-1-1/(2I_3\log l_0/z_0)}. \tag{8.7}$$

Since much of the heat injected into the flow is returned to the ground by transfer, the magnitude of the temperature changes is not established by the analysis but could be estimated by arguments similar to those used in § 5 for the wake of a fence.

For comparison, the temperature distribution for heat transfer related to temperature gradient by the mixing-length relation

$$\Phi_3 = \eta\phi_3'$$

is

$$\phi_3(\eta) = e^{-2\eta}, \tag{8.8}$$

again showing a restricted spread of heat compared with that of momentum in an analogous situation.

If the surface is impermeable to heat, a situation more likely to occur with diffusion of a pollutant than with heat, corresponding results can be obtained by putting Q_0 and θ_w equal to zero in the analysis, but it is advantageous to construct the line source by superimposing the distributions of surface flux

$$\left.\begin{matrix} Q(0) = 0, & x < 0, \\ Q(0) = Q_0, & x > 0, \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} Q(0) = 0, & x < \Delta, \\ Q(0) = -Q_0, & x > \Delta, \end{matrix}\right\}$$

where Δ is a very small length. The first distribution of surface flux causes a self-preserving temperature field

$$T = -(Q_0/ku_1)\,\phi_1(z/l_0), \tag{8.9}$$

and conservation of heat requires that

$$Q_0 x = \int_0^\infty UT\,dz = -\frac{Q_0 l_0}{k^2}\left\{\log\frac{l_0}{z_0}\int_0^\infty \phi_1(\eta)\,d\eta + \int_0^\infty \log\eta\,\phi_1(\eta)\,d\eta\right\}. \tag{8.10}$$

Equation (7.11) shows that

$$\int_0^\infty \phi_1(\eta)\,d\eta = -\tfrac{1}{2}, \tag{8.11}$$

and so

$$l_0\left\{\log l_0/z_0 - 2\int_0^\infty \phi_1(\eta)\log\eta\,d\eta\right\} = 2k^2 x, \tag{8.12}$$

in agreement with the asymptotic expression (4.8) for l_0. Superposition of the two distributions leads to the self-preserving field

$$T = \frac{Q_0\Delta}{ku_1}\frac{1}{l_0}\frac{dl_0}{dx}\,\eta\phi_1'(\eta), \tag{8.13}$$

where $Q_0\Delta$ is the strength of the line-source and $l_0^{-1}dl_0/dx$ is to be calculated from equation (8.12). The only restrictions placed on ϕ_1 are that it shall assume a logarithmic form for small η in the thermal equilibrium layer and be zero for large η. As a consequence of the restrictions on ϕ_1, the distribution function for the line-source $\psi_1(\eta) = \eta\phi_1'$ must be such that

$$\psi_1(0) = 1, \quad \int_0^\infty \psi_1(\eta)\,d\eta = \tfrac{1}{2}, \tag{8.14}$$

with l_0 defined by (8.12). In units of l_0, the centroid of the distribution is at

$$\bar{\eta} = 2\int_0^\infty \eta\psi_1(\eta)\,d\eta, \tag{8.15}$$

and, unless the vertical gradient of temperature changes sign,

$$\bar{\eta} \geqslant \tfrac{1}{4}. \tag{8.16}$$

The extreme value of $\frac{1}{4}$ is for a 'top-hat' distribution

$$\psi_1(\eta) = 1 \quad \text{for} \quad \eta < \tfrac{1}{2},$$
$$= 0 \quad \text{for} \quad \eta > \tfrac{1}{2}.$$

The actual centroid $\bar{z} = \bar{\eta} l_0$ is related to position by

$$\bar{z}\left(\log \bar{z}/z_0 - \log \bar{\eta} - 2\int_0^\infty \phi_1(\eta) \log \eta \, d\eta\right) = 2k^2\bar{\eta}x. \tag{8.17}$$

Ellison (1959) and Cermak (1963) have assumed Lagrangian similarity of the diffusion from a line source and derive equations for $\bar{z}$ very similar to (8.17) for large values of $\log l_0/z_0$ but with the right-hand side replaced by bkx where b is an undetermined constant equal to $2k\bar{\eta}$ in our notation. By deriving the equation from the step-flux solution, the constant $b = 2k\bar{\eta}$ is shown to be simply a function of the profile shape and not less than $\frac{1}{2}k$ whatever the profile.

Multipole sources can be treated in the same way, obtaining the temperature distributions by successive differentiations of the basic distribution with respect to x. The usefulness of these solutions is that the combination of a simple source with multipole sources permits the construction of special initial temperature distributions, including an elevated line source. The temperature scale of these distributions varies nearly as x^{-p} where p is the order of the multipole, i.e. 1 for a simple source 2 for a dipole, 3 for a quadrupole, etc.

9. Discussion

If self-preserving development of a flow is dynamically possible, it is commonly found that the corresponding real flow develops in a nearly self-preserving way, the only known exception being a cylinder wake formed in a stream undergoing uniform strain (Reynolds 1962; Keffer 1965), which is a very special case with two distinct sources of turbulent energy. As for boundary layers, the conditions for self-preserving development are satisfied only if the velocity variation is small except within the equilibrium layer at the surface. The condition is not very well satisfied in meteorological contexts, but the ratios of the terms in the equations of motion which are not self preserving to those that are vary very slowly, and the flow is almost self preserving in the sense that its development can run parallel with that of a truly self-preserving flow with nearly the same boundary conditions (Townsend 1961). Experience with boundary layers shows that accurate predictions can be obtained by assuming the velocity distribution to be self preserving and using the condition of total momentum to relate it to the boundary stresses. In a second paper, the technique is applied to the perturbation flows, and the improved approximations are discussed in relation to some of the observations.

If the surface roughness is uniform, the results of §§ 8 and 9 are equivalent to those obtained by Ellison (1959) and by Cermak (1963) using the concept of Lagrangian similarity of the diffusion process. Briefly, they assume that the

Lagrangian auto-correlation function of a particle takes a similarity form with a time scale that is proportional to the reciprocal of the rate of shear at the average height of the particles at the time. The assumption is equivalent to one of self preservation, but it is more difficult to isolate dynamical inconsistencies. In particular, it is clearly necessary that the stream velocity should be nearly constant for all the released particles, which is also a limitation of the self-preserving hypothesis. The theoretical advantage of using the Eulerian concept of self preservation is that the routine verification of dynamical consistency enables the inclusion of more complicated and realistic flows. A practical advantage is that actual flow modifications are as easily treated as diffusion of passive pollutants.

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Appendix: The dependence of the Reynolds stress on the velocity field

To illustrate the nature of the perturbation flows, velocity profiles have been derived assuming the equilibrium relation between Reynolds stress and local velocity gradient

$$\frac{kz}{\tau^{\frac{1}{2}}}\frac{\partial U}{\partial z} = 1,$$

but advection and convection of Reynolds stress are large enough to cause considerable departures from unity except within the equilibrium layer. If the flow is self-preserving, an improved and more plausible approximation to the true relation can be obtained. Assume structural similarity of the turbulent motion, i.e. that

(i) the ratio of the Reynolds stress to the kinetic energy of the 'active' turbulent motion is everywhere the same, and

(ii) the rate of energy dissipation is everywhere as it is in an equilibrium layer. The justification for the first assumption is that all the turbulence is generated or destroyed in shearing flow and remains as anisotropic as possible. The second one says that the effective scale of the turbulent motion is proportional to distance from the surface. Then the equation for the kinetic energy of the active component of the turbulence is

$$U\frac{\partial(\frac{1}{2}\overline{q^2})}{\partial x} + W\frac{\partial(\frac{1}{2}\overline{q^2})}{\partial z} - \tau\frac{\partial U}{\partial z} = -\frac{\tau^{\frac{3}{2}}}{kz} - \frac{\partial}{\partial z}(\overline{pw} + \tfrac{1}{2}\overline{q^2 w}). \tag{A 1}$$

As mentioned in §4, the equation takes a self-preserving form for large $\log l_0/z_0$ with

$$\tfrac{1}{2}\overline{q^2} = \tfrac{1}{2}\overline{q_1^2} + u_0 u_1 Q(\eta),$$

$$= \tfrac{1}{2}\overline{q_1^2}\left(1 + \frac{2u_0}{u_1}F(\eta)\right) \quad \text{by assumption (i)},$$

and

$$\tfrac{1}{2}\overline{q^2 w} + \overline{pw} = u_1^2 u_0 D(\eta),$$

$$\epsilon = \epsilon_1 + \frac{3u_1^2 u_0}{kl_0}\eta^{-1}F(\eta) \quad \text{by assumption (ii)}.$$

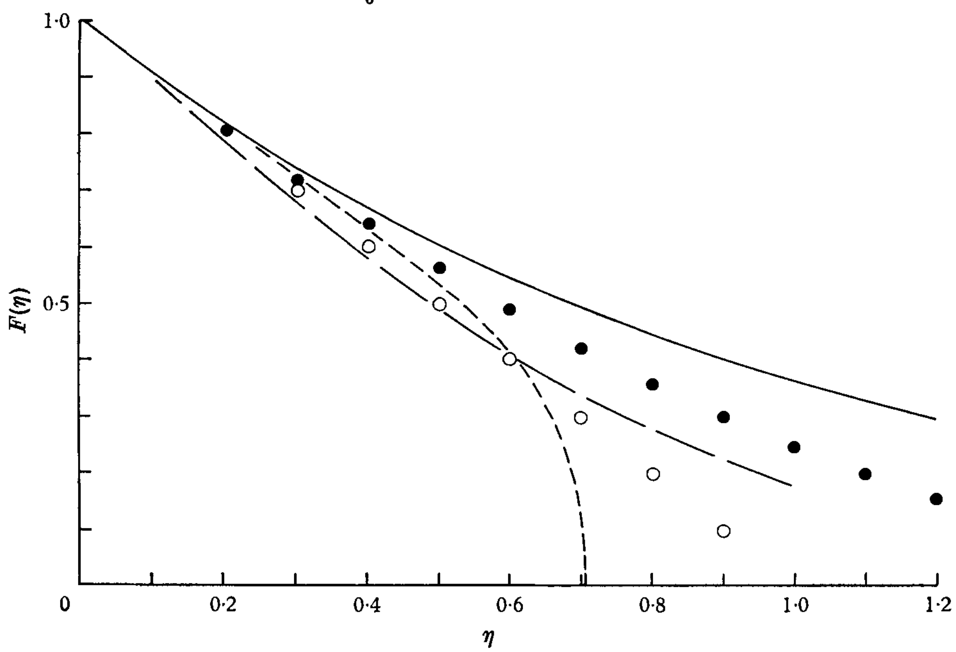

FIGURE 6. Comparison of stress distribution functions for a change of roughness. The curves represent solutions of equation (A 5): ——, mixing length, i.e. no advection or diffusion, $k^2\overline{q_1^2}/u_1^2 = 0$; – – –, with advection but without diffusion, i.e. $k^2\overline{q_1^2}/u_1^2 = 1$ and $\alpha = 0$; —— —, with advection and diffusion, i.e. $k^2\overline{q_1^2}/u_1^2 = \alpha = 1$. ●, Panofsky & Townsend profile; ○, Elliott profile.

After substitution of the self-preserving relations for dl_0/dx and du_0/dx, it takes the non-dimensional form

$$f' - \eta^{-1}F = kD' - 2k^2(\overline{q_1^2}/u_1^2)\,\eta F', \tag{A 2}$$

the terms on the right representing the effects of vertical diffusion and advection respectively. A crude assumption about the vertical diffusion is that

$$\tfrac{1}{2}\overline{q^2 w} + \overline{pw} = -\alpha k u_1 z\{\partial(\tfrac{1}{2}\overline{q^2})/\partial z\}, \tag{A 3}$$

implying that the diffusivity for energy is comparable with that for momentum in the undisturbed flow. In terms of the non-dimensional functions, it is

$$D(\eta) = -\alpha k(\overline{q_1^2}/u_1^2)\,\eta F',$$

and the energy equation becomes

$$nf' - F + k^2(\overline{q_1^2}/u_1^2)\{2\eta^2 F' + \alpha(\eta F' + F'')\} = 0. \tag{A 4}$$

Combined with the appropriate self-preserving relation between the velocity and stress distribution functions, a single equation for f or F is obtained which could be solved with the appropriate boundary conditions. For the change of roughness flow, equation (4.10) leads to

$$F + F' = k^2(\overline{q_1^2}/u_1^2)\{2\eta^2 F' + \alpha(\eta F' + \eta^2 F'')\}. \tag{A 5}$$

Remembering that the distribution without advection or diffusion is

$$F(\eta) = \exp(-\eta),$$

we can see that advection reduces the value of F and that diffusion opposes this tendency. For example, if $k^2(\overline{q^2}/u_1^2) = 1$ and $\alpha = 0$ (no diffusion),

$$F(\eta) = \left(\frac{1-\sqrt{2}\eta}{1+\sqrt{2}\eta}\right)^{1/(2\sqrt{2})} \quad \text{for} \quad \eta \leqslant \frac{1}{\sqrt{2}}.$$

On the other hand, if $\alpha = k^2(\overline{q^2}/u_1^2) = 1$, a very rough attempt at a numerical solution shows a partial return to the simple form. Figure 6 shows the stress distributions for the three transfer assumptions and also for the implied distributions of Elliott and of Panofsky & Townsend.